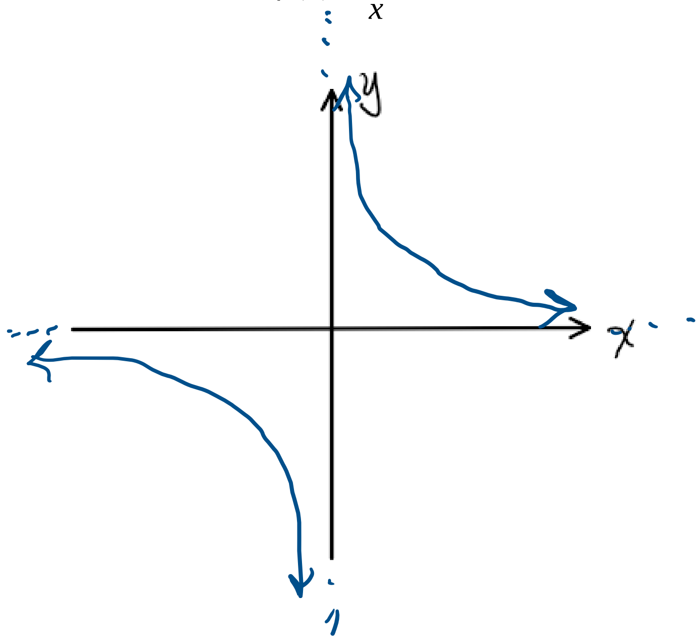
Introduction to Improper Integrals

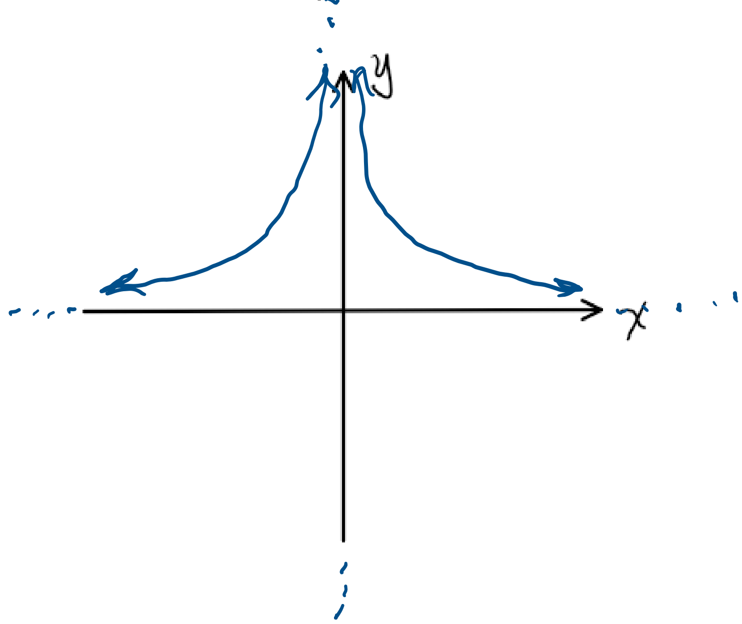
Integral is considered improper if:

- one (or both) bounds are +/- infinity
- integrand is discontinuous within interval (a,b)

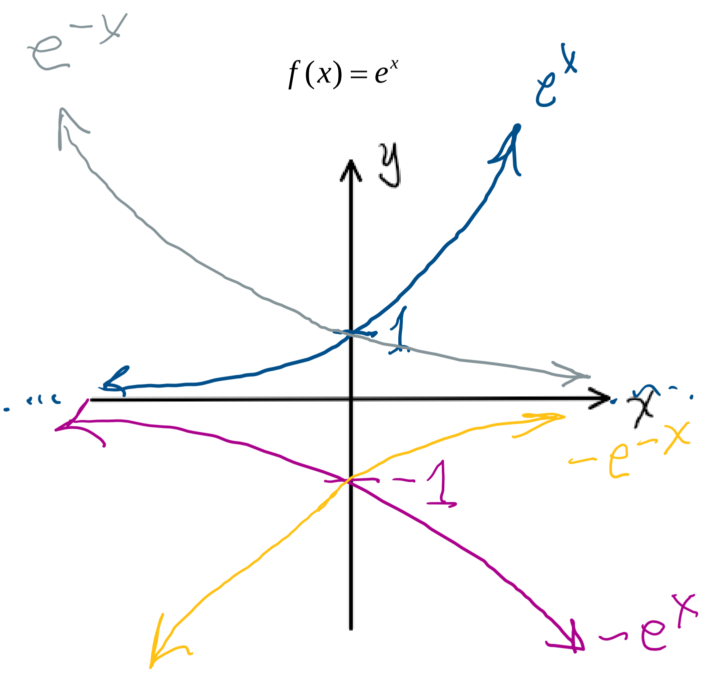
$f(x)=\dfrac{1}{x}$



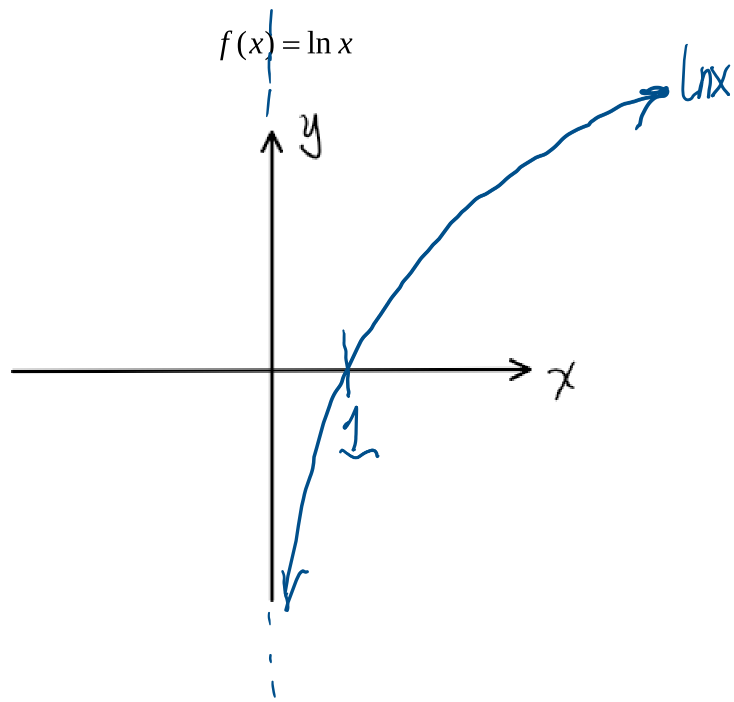
$f(x)=\dfrac{1}{x^2}$



$f(x)=e^x$



$f(x)=\ln x$



Definition:

integral is **convergent** if corresponding limit **exists**

or it is **divergent** if corresponding limit **does NOT exist (DNE)**

Recall:

$\int_1^{\infty} \frac{1}{x^2}\,dx$ converges to 1

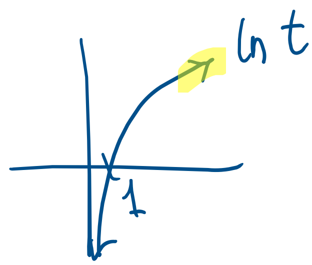
$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$
$$= \lim_{t\to\infty} \int_1^t x^{-2}\,dx$$
$$= \lim_{t\to\infty} \left.\frac{x^{-1}}{-1}\right|_1^t$$
$$= -\lim_{t\to\infty} \left.\frac{1}{x}\right|_1^t$$
$$= -\lim_{t\to\infty}\left(\frac{1}{t} - 1\right) = -\left(\cancel{\lim_{t\to\infty}\frac{1}{t}}^{\,0} - 1\right) = -(-1) = 1$$

$$\boxed{\therefore \int_1^{\infty} \frac{1}{x^2}\,dx \text{ converges to } 1}$$

$\int_1^{\infty} \frac{1}{x}\,dx$ diverges

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$
$$= \lim_{t\to\infty} \ln|x|\,\Big|_1^t$$
$$= \lim_{t\to\infty} \left(\ln|t| - \cancel{\ln 1}^{\,0}\right)$$
$$= \lim_{t\to\infty} \ln|t| = \infty \text{ or DNE}$$

$$\therefore \int_1^{\infty} \frac{1}{x}\,dx \text{ diverges}$$



Question: for what values of p is $\int_1^{\infty} \frac{1}{x^p}\,dx$ convergent?

$p=2$ $p=1$

already know that $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges and $\int_1^{\infty} \frac{1}{x^1}\,dx$ diverges

$\int_1^{\infty} \frac{1}{x^p}\,dx$
- converges when $p > 1$
- diverges when $p \le 1$

DO: ~~ex.~~ $\int_1^\infty \frac{1}{x^4}\,dx$ $= \lim_{t\to\infty}\int_1^t x^{-4}\,dx$

$= \lim_{t\to\infty} \left.\frac{x^{-3}}{-3}\right|_1^t$

$= -\frac{1}{3}\lim_{t\to\infty} \left.\frac{1}{x^3}\right|_1^t$

$= -\frac{1}{3}\left(\overbrace{\lim_{t\to\infty}\frac{1}{t^3}}^{0} - 1\right)$

$= -\frac{1}{3}(-1) \Rightarrow \boxed{\int_1^\infty \frac{1}{x^4}\,dx \text{ converges to } \frac{1}{3}}$

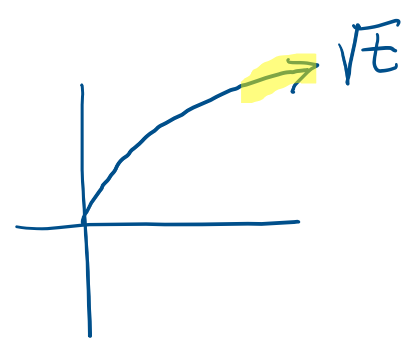
ex. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ $= \lim_{t\to\infty}\int_1^t \frac{1}{x^{1/2}}\,dx$ because $p = \frac{1}{2} < 1$, $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges

$= \lim_{t\to\infty}\int_1^t x^{-1/2}\,dx$

$= \lim_{t\to\infty} \left.\frac{x^{1/2}}{1/2}\right|_1^t$

$= 2\lim_{t\to\infty} \left.\sqrt{x}\right|_1^t$

$= \underbrace{2\left(\overbrace{\lim_{t\to\infty}\sqrt{t}}^{\infty} - 1\right)}_{\text{DNE}}$



bounds

Integrals with Discontinuities

ex. $\int_1^4 \frac{dx}{1-x} = \lim_{t\to 1^+} \int_t^4 \frac{1}{1-x}\,dx$

$\frac{1}{1-x}$ is UND @ $x=1$

$u = 1-x$

$du = -dx \Rightarrow -du = dx$

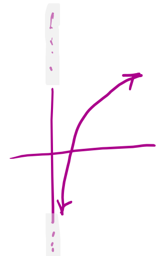
$= -\lim_{t\to 1^+} \int_{x=t}^{x=4} \frac{1}{u}\,du$

$= -\lim_{t\to 1^+} \ln|1-x| \Big|_t^4$

$= -\lim_{t\to 1^+} \left(\ln|1-4| - \ln|1-t|\right)$

$= -\left(\ln 3 - \lim_{t\to 1^+} \ln|1-t|\right)$ DNE

$= -\ln 3 +$



$\ln|1-1| = \ln|0|$

UND

$\boxed{\therefore \int_1^4 \frac{dx}{1-x} \text{ diverges}}$

ex. $\int_0^3 \frac{dx}{x-1}$

$\frac{1}{x-1}$ is UND @ $x=1$ split integral @ discontinuity

$= \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$

$= \underline{\lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx} + \lim_{s\to 1^+} \int_s^3 \frac{1}{x-1}\,dx$

$= \lim_{t\to 1^-} \ln|x-1| \Big|_0^t$

$= \lim_{t\to 1^-} \left(\ln|t-1| - \ln|0-1|\right)$

$= \lim_{t\to 1^-} \ln|t-1| - \cancel{\ln 1} \to 0$

$= \ln|1-1| = \ln 0$ DNE

$\therefore \boxed{\int_0^3 \frac{dx}{x-1} \text{ diverges}}$

LIATE

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

ex. $\int_0^1 \ln x \, dx = \lim_{t\to 0^+} \int_t^1 \ln x \, dx$ ← IBP

UND @ $x=0$

$= \lim_{t\to 0^+} \left(x\ln x \Big|_t^1 - \int_t^1 \cancel{x \cdot \frac{1}{x}}^{\,1} dx \right)$

$= \lim_{t\to 0^+} \left(\cancel{1\cdot \ln 1}^{\,0} - t\ln t - x \Big|_t^1 \right)$

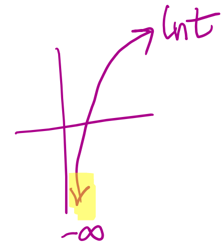
$= -\lim_{t\to 0^+} t \cdot \ln t - \left(1 - \cancel{\lim_{t\to 0^+} t}^{\,0}\right)$

$0\,(-\infty)$

indeterminate form

write $t\ln t$ as a fraction

$t = \frac{1}{\frac{1}{t}}$



$= -\lim_{t\to 0^+} \frac{\ln t^{\,-\infty}}{\frac{1}{t}_{\,\infty}} - 1$

L'H

negatives cancel

$= -\lim_{t\to 0^+} \boxed{\frac{\frac{1}{t}}{-\frac{1}{t^2}}} - 1$

$= \lim_{t\to 0^+} \cancel{t}^{\,0} - 1$

$= -1$

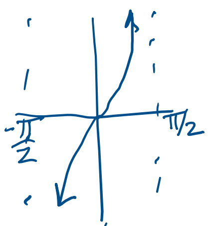
$$\boxed{\therefore \int_0^1 \ln x \, dx \text{ converges to } -1}$$

When Both Bounds Contain an Infinity

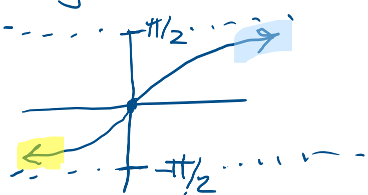
split into 2 integrals

ex. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

no x-values causes $\frac{1}{1+x^2}$ to be UND !!



y = tanx



y = arctanx

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{1}{1+x^2}\,dx + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t_1 \to -\infty} \arctan x \Big|_{t_1}^{0} + \lim_{t_2 \to \infty} \arctan x \Big|_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left(\cancelto{0}{\arctan 0} - \arctan t_1\right) + \lim_{t_2 \to \infty} \left(\arctan t_2 - \cancelto{0}{\arctan 0}\right)$$

$$= \underbrace{-\lim_{t_1 \to -\infty} \arctan t_1}_{= -\left(-\frac{\pi}{2}\right)} + \underbrace{\lim_{t_2 \to \infty} \arctan t_2}_{\frac{\pi}{2}}$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$\boxed{\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} \text{ converges to } \pi}$